Note: Your TA probably will not cover all the problems. This is totally fine, the discussion worksheets are deliberately made long so they can serve as a resource you can use to practice, reinforce, and build upon concepts discussed in lecture, readings, and the homework.

1 Asymptotics and Limits

If we would like to prove asymptotic relations instead of just using them, we can use limits.

Asymptotic Limit Rules: If $f(n), g(n) \geq 0$:

- If $\lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty$, then $f(n) = \mathcal{O}(g(n))$.
- If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = c$, for some $c > 0$, then $f(n) = \Theta(g(n))$.
- If $\lim_{n\to\infty} \frac{f(n)}{g(n)} > 0$, then $f(n) = \Omega(g(n))$.

Note that these are all sufficient (and not necessary) conditions involving limits, and are not true definitions of $\mathcal{O}$, Θ, and Ω. We highly recommend checking on your own that these statements are correct!)

(a) Prove that $n^3 = \mathcal{O}(n^4)$.

(b) Find an $f(n), g(n) \geq 0$ such that $f(n) = \mathcal{O}(g(n))$, yet $\lim_{n\to\infty} \frac{f(n)}{g(n)} \neq 0$.

(c) Prove that for any $c > 0$, we have $\log n = \mathcal{O}(n^c)$.

Hint: Use L'Hôpital's rule: If $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} g(n) = \infty$, then $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}$ (if the RHS exists)

(d) Find an $f(n), g(n) \geq 0$ such that $f(n) = \mathcal{O}(g(n))$, yet $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ does not exist. In this case, you would be unable to use limits to prove $f(n) = \mathcal{O}(g(n))$.
Hint: think about oscillating functions!

2 Recurrence Relations

Solve the following recurrence relations, assuming base cases $T(0) = T(1) = 1$:

(a) $T(n) = 2 \cdot T(n/2) + O(n)$

(b) $T(n) = 2 \cdot T(n/2) + O(n \log n)$

(c) $T(n) = T(n-1) + n$

(d) $T(n) = 3 \cdot T(n-2) + 5$

(e) $T(n) = 3T(n^{1/3}) + O(\log n)$

(f) $T(n) = T(n-1) + T(n-2)$